

On trigonometric intertwining vectors and non-dynamical R -matrix for the Ruijsenaars model

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Abstract

We elaborate the trigonometric version of intertwining vectors and factorized L -operators. The starting point is the corresponding elliptic construction with Belavin's R -matrix. The naive trigonometric limit is singular and a careful analysis is needed. It is shown that the construction admits several different trigonometric degenerations. As a by-product, a quantum Lax operator for the trigonometric Ruijsenaars model intertwined by a non-dynamical R -matrix is obtained. The latter differs from the standard trigonometric R -matrix of A_n type. A connection with the dynamical R -matrix approach is discussed.

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1 Introduction

Recently, a new type of quantum Lax operator was suggested [1] for the elliptic Ruijsenaars model [2]. In contrast to the previous versions, this L -operator obeys the standard " $RLL = LLR$ " relation with an R -matrix that does not depend on dynamical variables. Specifically, it is Belavin's elliptic $n^2 \times n^2$ R -matrix [3], where n equals the number of particles in the model. The construction relies upon the technique of intertwining vectors and factorized L -operators [4], [5], [6].

The trigonometric degeneration of this construction is not automatic since the elliptic intertwining vectors diverge as the elliptic nome tends to zero. To get a proper analog, one should either construct trigonometric intertwining vectors independently or apply a gauge transformation before taking the limit.

This paper deals with that trigonometric limit. As a result, we get the quantum Lax operator for the trigonometric Ruijsenaars model which obeys the R -matrix quadratic algebra with a *non-dynamical* trigonometric R -matrix. Remarkably, this R -matrix *differs* from the standard one [7]. In an implicit form the same result was obtained in [8] by means of the *dynamical* R -matrix approach.

In fact, the trigonometric case allows for different versions of the intertwining vectors that, therefore, leads to different types of quantum L -operators and R -matrices for the same trigonometric Ruijsenaars model. Among them, there is a version without a spectral parameter, which coincides with the Cremmer-Gervais R -matrix [9].

Let us give an example of the non-standard trigonometric R -matrix for $n = 2$. We get it as a twisted degeneration of Baxter's 4×4 R -matrix [10]. It has the form

$$R(u) = \begin{pmatrix} \sin \pi(u + 2\eta) & 0 & 0 & 0 \\ 0 & \sin \pi u & \sin 2\pi\eta & 0 \\ 0 & \sin 2\pi\eta & \sin \pi u & 0 \\ \alpha \sin \pi u \sin \pi(u + 2\eta) & 0 & 0 & \sin \pi(u + 2\eta) \end{pmatrix}, \quad (1.1)$$

where α is an arbitrary constant. This R -matrix satisfies the Yang-Baxter equation for any α . If $\alpha \neq 0$, we can set $\alpha = 1$ applying a constant gauge transformation. At $\alpha = 0$ we get the ordinary 6-vertex trigonometric R -matrix. By analogy, one may introduce the *7-vertex* lattice statistical model using matrix elements of (1.1) as Boltzmann weights [11], [12]. For periodic boundary conditions the partition function of the 7-vertex model coincides with that of the 6-vertex one. The R -matrix (1.1) appeared for the first time in [13]. Here we present explicit formulas for non-standard A_{n-1} trigonometric R -matrices with spectral parameter for $n \geq 3$.

Particular cases of these R -matrices were already mentioned in the literature [14], [8]. In [14], they were derived as reductions of the infinite dimensional R -matrix with complete \mathbf{Z} -symmetry (without any relation to the Ruijsenaars model). In [8], the "modified" commutation relation " $R^*LL = LLR$ " for the trigonometric Ruijsenaars model was pointed out. The L -operator was obtained from the dynamical one by a gauge transformation depending on dynamical variables. However, the symmetry $R = R^*$ of the R -matrix leading to the ordinary relation " $RLL = LLR$ " was not noticed there.

Matrix elements of L -operators intertwined by the R -matrix (1.1) obey a quadratic algebra that can be obtained as a degeneration of the Sklyanin algebra. In fact, this algebra lies "in between" the q -deformed universal enveloping $U_q(sl(2))$ with $q = e^{2i\pi\eta}$ and the Sklyanin algebra [15]. Representations on this algebra realized by difference operators were studied in [16]. We will see that the same realization is reproduced by means of trigonometric intertwining vectors.

Among other applications of the non-standard trigonometric R -matrices we point out their relation to statistical models of the IRF (interaction round a face) type, in particular, to "solid-on-solid" (SOS) models. It turns out that the non-standard trigonometric R -matrix is involved in the vertex-face correspondence with the trigonometric SOS model. This can be seen by constructing intertwining vectors which are obtained as a special trigonometric limit of the elliptic ones. In the elliptic case, the vertex-face correspondence was first established by R.Baxter in [10] for the eight-vertex model ($n = 2$) and by M.Jimbo, T.Miwa and M.Okado [17] for general A_{n-1} -type models.

Introducing trigonometric intertwining vectors, we then make the *factorized L-operator* out of them. L -operators of such a kind first appeared in [18] for a particular case. Later, they were used for the chiral Potts model and its generalizations [19]. The elliptic version of factorized L -operators was found in [4], [5] and [6].

The commuting integrals of motion (IM) associated with this L -operator coincide with Hamiltonian of the trigonometric Ruijsenaars model. The first non-trivial Hamiltonian is obtained by taking trace of the factorized L -operator. Thus, we can refer to this L -operator as the quantum Lax operator for the trigonometric Ruijsenaars model. This gives a possibility to use the machinery of quantum inverse scattering method [20]. We emphasize this because other versions of the trigonometric Ruijsenaars model L -operator satisfies the *modified* Yang-Baxter equation [21], [22] with *dynamical* R -matrix [23], [8]. These two versions of the trigonometric Ruijsenaars model L -operators differ by a *dynamical* gauge transformation [8].

This observation might clarify the nature of dynamical R -matrices. We see that there are two R -matrix approaches to the model: the first one is based on dynamical R -matrix [8] and the second one uses the ordinary (non-dynamical) one. We demonstrate that they lead to the same results.

Note, that another version of quantum elliptic L -operator obtained from cotangent bundles construction was proposed in [24]. In fact, it was connected with factorized elliptic L -operator [1] by a twist transformation with intertwining vectors.

The structure of the paper is as follows. Sect. 2 contains a detailed analysis of the A_1 -case (2×2 L -operators). This corresponds to the 2-particle Ruijsenaars model. Some interesting algebraic structures related to the Sklyanin algebra are discussed here. Starting from Sect. 3 we deal with the general A_{n-1} -case (n -particle Ruijsenaars model). In Sect. 3 different trigonometric R -matrices are obtained as certain degenerations of the elliptic Belavin R -matrix. Different versions of the trigonometric vertex-face correspondence are discussed in Sect. 4. In Sect. 5 the factorized L -operator for the trigonometric Ruijsenaars model is constructed. The connection with dynamical R -matrices is discussed in Sect. 6. The Appendices contain some technical remarks.

2 The twisted trigonometric limit of Baxter's R -matrix and the 7-vertex model

In this section we discuss a non-standard trigonometric degeneration of elliptic R -matrices for the simplest example of Baxter's R -matrix corresponding to the 8-vertex model.

The universal elliptic L -operator with 2-dimensional auxiliary space has the form

$$L(u) = \sum_{a=0}^3 W_a(u) S_a \otimes \sigma_a. \quad (2.1)$$

Here $W_a(u) = W_a(u|\eta, \tau)$, $a = 0, \dots, 3$ are functions of the variable u (called the spectral parameter) with parameters η and τ :

$$W_a(u) = \frac{\theta_{\iota(a)}(u)}{\theta_{\iota(a)}(\eta)}, \quad \iota(a) = a + (-1)^a \quad (2.2)$$

($\theta_a(x) = \theta_a(x|\tau)$ are standard Jacoby theta-functions with characteristics and the modular parameter τ); σ_α are Pauli matrices (σ_0 is the unit matrix). The operators S_0, S_α , $\alpha = 1, 2, 3$, obey the Sklyanin algebra [15]:

$$\begin{aligned} [S_0, S_\alpha]_- &= iJ_{\beta\gamma}[S_\beta, S_\gamma]_+, \\ [S_\alpha, S_\beta]_- &= i[S_0, S_\gamma]_+ \end{aligned} \quad (2.3)$$

($[A, B]_\pm = AB \pm BA$, a triple of Greek indices α, β, γ in (2.3) stands for any *cyclic* permutation of $(1, 2, 3)$). Structure constants of this algebra $J_{\alpha\beta}$ have the form

$$J_{\alpha\beta} = \frac{J_\beta - J_\alpha}{J_\gamma},$$

where

$$J_\alpha = \frac{\theta_{\iota(\alpha)}(2\eta)\theta_{\iota(\alpha)}(0)}{\theta_{\iota(\alpha)}^2(\eta)}$$

Relations (2.3) were introduced by E.Sklyanin as the minimal set of conditions under which the L -operators (2.1) satisfy the equation

$$R_{12}^{(ell)}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}^{(ell)}(u-v), \quad (2.4)$$

crucial for the solvability by the algebraic Bethe ansatz method. Baxter's R -matrix $R^{(ell)}(u)$,

$$R^{(ell)}(u) = \sum_{a=0}^3 W_a(u+\eta)\sigma_a \otimes \sigma_a, \quad (2.5)$$

is the elliptic solution of the quantum Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \quad (2.6)$$

In (2.4), (2.6) we use the following standard notation: $L_1 \equiv L \otimes I$, $L_2 \equiv I \otimes L$, (I is the identity operator) The R -matrix R_{12} acts in the tensor product $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ as $R(u)$ on the first and the second spaces and as the identity operator on the third one (similarly for R_{13} , R_{23}).

Explicitly, the R -matrix (2.5) reads

$$R^{(ell)}(u) = \begin{pmatrix} a^{(ell)} & 0 & 0 & d^{(ell)} \\ 0 & b^{(ell)} & c^{(ell)} & 0 \\ 0 & c^{(ell)} & b^{(ell)} & 0 \\ d^{(ell)} & 0 & 0 & a^{(ell)} \end{pmatrix} \quad (2.7)$$

(up to a common constant factor),

$$\begin{aligned} a^{(ell)} &= \theta_0(2\eta|2\tau)\theta_0(u|2\tau)\theta_1(u+2\eta|2\tau), \\ b^{(ell)} &= \theta_0(2\eta|2\tau)\theta_1(u|2\tau)\theta_0(u+2\eta|2\tau), \\ c^{(ell)} &= \theta_1(2\eta|2\tau)\theta_0(u|2\tau)\theta_0(u+2\eta|2\tau), \\ d^{(ell)} &= \theta_1(2\eta|2\tau)\theta_1(u|2\tau)\theta_1(u+2\eta|2\tau). \end{aligned} \quad (2.8)$$

We are going to study the limit when the elliptic parameter $h = e^{\pi i \tau}$ tends to 0. The matrix elements have the following behavior as $h \rightarrow 0$:

$$\begin{aligned} a^{(ell)} &= 2h^{1/2} \sin \pi(u+2\eta) + O(h^{5/2}), \\ b^{(ell)} &= 2h^{1/2} \sin \pi u + O(h^{5/2}), \\ c^{(ell)} &= 2h^{1/2} \sin 2\pi\eta + O(h^{5/2}), \\ d^{(ell)} &= 8h^{3/2} \sin 2\pi\eta \sin \pi u \sin \pi(u+2\eta) + O(h^{11/2}). \end{aligned} \quad (2.9)$$

The limit $h \rightarrow 0$ yields the standard trigonometric R -matrix.

One may apply an h -dependent gauge transformation before taking the limit. Let us choose the matrix of the gauge transformation in the form

$$G = \begin{pmatrix} h^{1/4}\gamma^{-1/2} & 0 \\ 0 & h^{-1/4}\gamma^{1/2} \end{pmatrix} \quad (2.10)$$

with a parameter γ . This transformation becomes singular at the limiting point $h = 0$, so the result of the limit appears to be different (though trigonometric) from the standard trigonometric R -matrix. We

set $G_1 = G \otimes I$, $G_2 = I \otimes G$, as usual, so $G_1 G_2 = \text{diag}(h^{1/2}\gamma^{-1}, 1, 1, h^{-1/2}\gamma)$. The gauge-transformed R -matrix is $R_h(u) = G_1 G_2 R^{(ell)}(u) (G_1 G_2)^{-1}$. We define

$$R(u) = \frac{1}{2} \lim_{h \rightarrow 0} h^{-1/2} R_h(u). \quad (2.11)$$

The result is (up to an irrelevant common factor)

$$R(u) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}, \quad (2.12)$$

where

$$\begin{aligned} a &= \sin \pi(u + 2\eta), \\ b &= \sin \pi u, \\ c &= \sin 2\pi\eta, \\ d &= 4\gamma^2 \sin 2\pi\eta \sin \pi u \sin \pi(u + 2\eta). \end{aligned} \quad (2.13)$$

The corresponding gauge transformation of the L -operator is $L(u) \rightarrow GL(u)G^{-1}$. In order to find its explicit limiting form, it is necessary to fix the behavior of the operators S_a as $h \rightarrow 0$. Using results of the paper [16], we define generators A, B, C, D of the degenerate Sklyanin algebra by extracting singular terms in the expansion of S_a near $h = 0$:

$$\begin{aligned} A &= -\frac{h^{-1/2}}{2 \sin 2\pi\eta} (\cos \pi\eta S_0 + i \sin \pi\eta S_3), \\ D &= -\frac{h^{-1/2}}{2 \sin 2\pi\eta} (\cos \pi\eta S_0 - i \sin \pi\eta S_3), \\ C &= -\frac{h^{1/2}}{2 \sin 2\pi\eta} (S_1 - i S_2), \\ B &= -\frac{h^{-3/2}}{8 \sin 2\pi\eta} (S_1 + i S_2). \end{aligned} \quad (2.14)$$

From now on we put $\gamma = 1$ without loss of generality. With this definition we obtain at $h = 0$ the following trigonometric L -operator:

$$L(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}, \quad (2.15)$$

where

$$\begin{aligned} \mathcal{A}(u) &= e^{i\pi u} A - e^{-i\pi u} D, \\ \mathcal{B}(u) &= i \sin 2\pi\eta C, \\ \mathcal{C}(u) &= 2i \sin 2\pi\eta [2B - (\cos 2\pi u - \cos 2\pi\eta)C], \\ \mathcal{D}(u) &= e^{i\pi u} D - e^{-i\pi u} A. \end{aligned} \quad (2.16)$$

The generators A, B, C, D satisfy the quadratic algebra [16]:

$$DC = e^{2\pi i\eta} CD, \quad CA = e^{2\pi i\eta} AC,$$

$$\begin{aligned}
AD - DA &= -2i \sin^3 2\pi\eta C^2, \\
BC - CB &= \frac{A^2 - D^2}{2i \sin 2\pi\eta}, \\
AB - e^{2\pi i\eta} BA &= e^{2\pi i\eta} DB - BD = \frac{i}{2} \sin 4\pi\eta (CA - DC).
\end{aligned} \tag{2.17}$$

The Casimir elements are

$$\begin{aligned}
\Omega_0 &= e^{2\pi i\eta} AD - \sin^2 2\pi\eta C^2, \\
\Omega_1 &= \frac{e^{-2\pi i\eta} A^2 + e^{2\pi i\eta} D^2}{4 \sin^2 2\pi\eta} - BC - \cos 2\pi\eta C^2.
\end{aligned} \tag{2.18}$$

Similar degenerations of elliptic quadratic algebras were considered in [25].

An important class of representations of this algebra is given by the following explicit construction. Let $\Phi(u)$ denote the c -number matrix

$$\Phi(u) = \begin{pmatrix} 1 & 1 \\ 2 \cos \pi(u - 2\lambda_1) & 2 \cos \pi(u - 2\lambda_2) \end{pmatrix}, \tag{2.19}$$

where λ_1, λ_2 are complex parameters. The factorized L -operator can be written as

$$L^{(F)}(u) =: \Phi(u + 2\ell\eta) e^{\eta\sigma_3(\partial_{\lambda_1} - \partial_{\lambda_2})} \Phi^{-1}(u - 2\ell\eta) :, \tag{2.20}$$

where ℓ is a constant, $\sigma_3 = \text{diag}(1, -1)$ and the normal ordering $::$ means that the operators ∂_{λ_i} should be moved to the right after performing the matrix product. Explicitly, acting to functions of λ_1, λ_2 , we have:

$$\begin{aligned}
&L_{ij}^{(F)}(u) f(\lambda_1, \lambda_2) \\
&= \Phi_{i1}(u + 2\ell\eta) [\Phi^{-1}(u - 2\ell\eta)]_{1j} f(\lambda_1 + \eta, \lambda_2 - \eta) \\
&+ \Phi_{i2}(u + 2\ell\eta) [\Phi^{-1}(u - 2\ell\eta)]_{2j} f(\lambda_1 - \eta, \lambda_2 + \eta).
\end{aligned}$$

It can be shown that for any ℓ the L -operator (2.20) satisfies the intertwining relation (2.4) with the R -matrix (2.12).

Therefore, we get a family of representations of the algebra (2.17) realized in the space of functions $f(\lambda_1, \lambda_2)$. Comparing (2.16) and (2.20), we identify

$$L^{(F)}(u + \lambda_1 + \lambda_2) = L(u), \tag{2.21}$$

where $L(u)$ is as in (2.15), (2.16) up to an irrelevant common factor. This identification gives the following realization of the algebra (2.17) by difference operators:

$$\begin{aligned}
A &= \frac{e^{-2\pi i\eta\ell}}{\sin \pi\lambda_{21}} (e^{-\pi i\lambda_{21}} T - e^{\pi i\lambda_{21}} T^{-1}), \\
B + \frac{1}{2} \cos 2\pi\eta C &= \frac{1}{2i \sin 2\pi\eta \sin \pi\lambda_{21}} (\cos 2\pi(\lambda_{21} + 2\ell\eta)T - \cos 2\pi(\lambda_{21} - 2\ell\eta)T^{-1}), \\
C &= -\frac{1}{i \sin 2\pi\eta \sin \pi\lambda_{21}} (T - T^{-1}), \\
D &= \frac{e^{2\pi i\eta\ell}}{\sin \pi\lambda_{21}} (e^{-\pi i\lambda_{21}} T^{-1} - e^{\pi i\lambda_{21}} T).
\end{aligned} \tag{2.22}$$

Here $T^{\pm 1} f(\lambda_1, \lambda_2) = f(\lambda_1 \pm \eta, \lambda_2 \mp \eta)$, $\lambda_{21} \equiv \lambda_2 - \lambda_1$.

Note that T commutes with functions of $\lambda_1 + \lambda_2$ that allows one to consider $\lambda_1 + \lambda_2$ as a constant including it in the spectral parameter. After evident redefinitions, the realization (2.22) coincides with the

one given in [16]. The parameter ℓ is called spin of the representation. If ℓ is a positive integer or half-integer, there is a $(2\ell + 1)$ -dimensional invariant subspace spanned by symmetric Laurent polynomials of degree 2ℓ .

Taking trace of the L -operator (2.21), we get, up to a common u -dependent factor, the operator

$$\hat{H} = \frac{\sin \pi (\lambda_{21} + \ell\eta)}{\sin \pi \lambda_{21}} e^{\eta(\partial_{\lambda_1} - \partial_{\lambda_2})} + \frac{\sin \pi (\lambda_{21} - \ell\eta)}{\sin \pi \lambda_{21}} e^{-\eta(\partial_{\lambda_1} - \partial_{\lambda_2})}, \quad (2.23)$$

which is "gauge equivalent" to the Hamiltonian of the trigonometric 2-particle Ruijsenaars model.

Let us consider two important limits of the factorized L -operator (2.20). One of them is to replace $\lambda_k \rightarrow \lambda_k + i\Lambda$ and after that let $\Lambda \rightarrow +\infty$ simultaneously with the gauge transformation $L(u) \rightarrow g(u)L(u)g^{-1}(u)$, where $g(u)$ is the diagonal matrix

$$g(u) = \text{diag} \left(\exp\left[\frac{i\pi}{2}(u - 2i\Lambda)\right], \exp\left[-\frac{i\pi}{2}(u - 2i\Lambda)\right] \right).$$

In this way we get the u -independent L -operator

$$L = \frac{1}{\sin \pi \lambda_{21}} \begin{pmatrix} e^{-2i\pi(\lambda_2 + \ell\eta)}T - e^{-2i\pi(\lambda_1 + \ell\eta)}T^{-1} & T^{-1} - T \\ e^{-2i\pi(\lambda_1 + \lambda_2)}(T - T^{-1}) & e^{-2i\pi(\lambda_2 - \ell\eta)}T^{-1} - e^{-2i\pi(\lambda_1 - \ell\eta)}T \end{pmatrix} \quad (2.24)$$

This L -operator is intertwined by the trigonometric R -matrix

$$R'(u) = \begin{pmatrix} \sin \pi(u + 2\eta) & 0 & 0 & 0 \\ 0 & \sin \pi u & e^{-\pi i u} \sin 2\pi\eta & 0 \\ 0 & e^{\pi i u} \sin 2\pi\eta & \sin \pi u & 0 \\ 0 & 0 & 0 & \sin \pi(u + 2\eta) \end{pmatrix}, \quad (2.25)$$

i.e., it holds

$$R'_{12}(u - v)L_1L_2 = L_2L_1R'_{12}(u - v). \quad (2.26)$$

Note that the R -matrix *depends* on the spectral parameter while the L -operators do not. Tending $u - v \rightarrow i\infty$, we get the version of these commutation relations without spectral parameter.

Let us stress that $\text{tr}L$ again yields the trigonometric Ruijsenaars Hamiltonian (2.23). At the same time commutation relations (2.26) define the algebra of functions $\text{Fun}_q(SL(2))$ on the quantum group $SL(2)$ with $q = \exp(2\pi i\eta)$. The L -operator (2.24) provides a representation of this algebra. Therefore, the 2-particle trigonometric Ruijsenaars model appears to be connected with representation of the algebra $\text{Fun}_q(SL(2))$.

Another limit yields the L -operator constructed from representations of the *dual algebra* to the quantum algebra of functions $\text{Fun}_q(SL(2))$, the q -deformation of the universal enveloping of the $sl(2)$ algebra, $U_q(sl(2))$. The limit consists in replacing $\lambda_1 \rightarrow \lambda_1 - i\Lambda$, $\lambda_2 \rightarrow \lambda_2 + i\Lambda$ simultaneously with a proper gauge transformation such that the limit $\Lambda \rightarrow \infty$ is finite. In this way we get the L -operator

$$L(u) \rightarrow \begin{pmatrix} e^{i\pi(u - 2\ell\eta)}T - e^{-i\pi(u - 2\ell\eta)}T^{-1} & e^{i\pi\lambda_{21}}(T^{-1} - T) \\ e^{-i\pi\lambda_{21}}(e^{-4i\pi\ell\eta}T - e^{4i\pi\ell\eta}T^{-1}) & e^{i\pi(u + 2\ell\eta)}T^{-1} - e^{-i\pi(u + 2\ell\eta)}T \end{pmatrix}. \quad (2.27)$$

This L -operator is intertwined by the standard trigonometric R -matrix. Its trace is an operator with constant coefficients.

The fundamental representation of the algebra (2.17) acts in the 2-dimensional invariant subspace for $\ell = \frac{1}{2}$. Explicitly, it is provided by

$$L(u) = R(u - \eta). \quad (2.28)$$

Let us introduce a lattice statistical model using the R -matrix (2.12) as the matrix of Boltzmann weights corresponding to different configurations of arrows around a vertex. We call it *7-vertex model*

because there are 7 non-zero weights. Note that the matrix elements a, b, c, d are independent Boltzmann weights. They always can be parameterized as in (2.13). The 6-vertex model is reproduced at $\gamma = 0$.

The monodromy matrix is constructed in the standard way:

$$\mathcal{T}(u) = R_{0N}(u) \dots R_{02}(u) R_{01}(u) \quad (2.29)$$

(the auxiliary "horizontal" space is labeled by 0). The transfer matrix is $T(u) = \text{tr}_0 \mathcal{T}(u)$, where trace is taken in the auxiliary space. It follows from the Yang-Baxter equation that $[T(u), T(v)] = 0$. The partition function for the toroidal $M \times N$ lattice is $\mathcal{Z} = \text{Tr } T^M(u)$ (here the trace is taken in the tensor product of vertical spaces).

It is well known that in any configuration of arrows on the toroidal lattice the d -vertices always come in pairs with the d' -vertices (in which all arrows are reversed), so the partition function depends on $(dd')^{1/2}$ only. In our case this means that the partition function does not depend on γ , i.e. it is the same as that of the standard 6-vertex model. However, the eigenvectors of the transfer matrix are different. In particular, for non-zero γ the transfer matrix is not completely diagonalizable, i.e., it contains Jordan cells. Since the number of vertices looking up and down along the row is not conserved from row to row, the usual Bethe ansatz technique for finding eigenvectors is not applicable. One should apply Baxter's method used in the 8-vertex model. Whence the 7-vertex case might serve as a toy model of Baxter's elliptic construction.

3 Non-standard trigonometric R -matrices for $n \geq 3$

In this section we obtain the non-standard trigonometric degenerations of the elliptic Belavin R -matrix [3]. To write it down explicitly, we need theta functions with rational characteristics

$$\theta^{(j)}(u) = \sum_{m \in \mathbf{Z}} \exp \left[\pi \sqrt{-1} n \tau \left(m + \frac{1}{2} - \frac{j}{n} \right)^2 + 2\pi \sqrt{-1} \left(m + \frac{1}{2} - \frac{j}{n} \right) \left(u + \frac{1}{2} \right) \right]. \quad (3.1)$$

(Here and below we write $\sqrt{-1}$ instead of the imaginary unit to avoid coincidence with the index label.) Choosing the standard bases $(E_{ii'})_{kl} = \delta_{ik} \delta_{i'l}$ in the space of $n^2 \times n^2$ matrices, we write the Belavin R -matrix in the form

$$R^{(ell)}(u) = \sum_{i', j'=1}^n R^{(ell)}(u)_{i'j'}^{ij} E_{ii'} \otimes E_{jj'},$$

where the matrix elements are [26]:

$$R^{(ell)}(u)_{i'j'}^{ij} = \delta_{i+j, i'+j' \bmod n} \frac{\theta^{(i'-j')}(u+2\eta) \theta^{(0)}(u)}{\theta^{(i'-i)}(2\eta) \theta^{(i-j')}(u)}. \quad (3.2)$$

The normalization here is different from the one used in the previous section for $n = 2$.

As in the case $n = 2$, we consider the limit when the elliptic nome $h = e^{\pi i \tau}$ tends to 0. The theta functions $\theta^{(j)}(u)$ behave as

$$\begin{aligned} \theta^{(j)}(u) &= \exp \left[2\pi \sqrt{-1} \left(\frac{1}{2} \text{sign } j - \frac{j}{n} \right) \left(u + \frac{1}{2} \right) \right] h^{n(\frac{1}{2} - \frac{|j|}{n})^2} + O \left(h^{n(\frac{1}{2} + \frac{|j|}{n})^2} \right), \quad -n+1 \leq j \leq n-1, \quad j \neq 0, \\ \theta^{(0)}(u) &= -2 \sin \pi u h^{n/4} + O \left(h^{\frac{9}{4}n} \right). \end{aligned}$$

So the asymptotic of the Belavin R -matrix $R^{(ell)}(u)_{i'j'}^{ij}$ can be written as follows:

$$R^{(ell)}(u)_{i'j'}^{ij} = \delta_{i+j, i'+j' \bmod n} O \left(h^{\frac{1}{n} [2(i-j')(i'-i) + n|i'-i| + n|i-j'| - n|i'-j'|]} \right), \quad h \rightarrow 0. \quad (3.3)$$

Taking the limit $h \rightarrow 0$, we get the standard trigonometric A_{n-1} R -matrix [7]

$$R_{A_{n-1}}(u)_{i'j'}^{ij} = \delta_{i,j} \delta_{i,i'} \delta_{i',j'} \frac{\sin \pi(u+2\eta)}{\sin 2\pi\eta}$$

$$\begin{aligned}
& + \delta_{i,i'} \delta_{j,j'} \varepsilon(i' \neq j') \frac{\sin \pi u}{\sin 2\pi \eta} \exp \left[\frac{2\pi \sqrt{-1} \eta}{n} (2(j-i) - n \operatorname{sign}(j-i)) \right] \\
& + \delta_{i,j} \delta_{i',j'} \varepsilon(i' \neq j') \exp \left[-\frac{\pi \sqrt{-1} u}{n} (2(j-i) - n \operatorname{sign}(j-i)) \right].
\end{aligned} \tag{3.4}$$

Here, $\varepsilon(\text{condition})$ is equal to 1 if the condition is true 0 otherwise.

As in Sect. 2, we may apply an h -dependent gauge transformation before the limit $h \rightarrow 0$. The matrix of gauge transformation has the form

$$G_{ij} = \delta_{ij} h^{n(\frac{1}{2} - \frac{i}{n})^2}, \quad i, j = 1, \dots, n. \tag{3.5}$$

The result of the limit differs from the standard trigonometric A_{n-1} R -matrix because the gauge transformation is singular at the point $h = 0$.

We are interested in the non-standard trigonometric limit

$$\tilde{R}(u) = \lim_{h \rightarrow 0} G_1 G_2 R^{(ell)}(u) G_1^{-1} G_2^{-1}, \tag{3.6}$$

where $G_1 = G \otimes I$, $G_2 = I \otimes G$.

Matrix elements of the Belavin R -matrix 3.2) are non-zero provided $i+j = i'+j'+M$ for $M = -n, 0, n$. It is not difficult to see that the leading terms of the gauge transformed elliptic R -matrix as $h \rightarrow 0$ are

$$(G_1 G_2 R_B(u) G_1^{-1} G_2^{-1})_{i'j'}^{ij} = \delta_{i+j, i'+j' \bmod n} O\left(h^{\frac{M}{n}(2j-M) - M + |i'-i| + |i-j'| + |i'-j'|}\right). \tag{3.7}$$

Extracting terms of zero degree in h , we get the non-standard trigonometric R -matrix (3.6):

$$\tilde{R}(u)_{i'j'}^{ij} = R_{A_{n-1}}(u)_{i'j'}^{ij} + S(u)_{i'j'}^{ij}, \tag{3.8}$$

where

$$\begin{aligned}
S(u)_{i'j'}^{ij} = & - 2\sqrt{-1} \sin \pi u \delta_{i+j, i'+j'} \varepsilon(i' < i < j') \exp \left[\frac{2\pi \sqrt{-1}}{n} ((i-i')u + 2(j'-i)\eta) \right] \\
& + 2\sqrt{-1} \sin \pi u \delta_{i+j, i'+j'} \varepsilon(i' > i > j') \exp \left[\frac{2\pi \sqrt{-1}}{n} ((i-i')u + 2(j'-i)\eta) \right] \\
& - 2\sqrt{-1} \sin \pi u \delta_{i+j, i'+j'-n} \delta_{i', n} \varepsilon(j' < n) \exp \left[\frac{2\pi \sqrt{-1}}{n} (iu + 2j\eta) \right] \\
& + 2\sqrt{-1} \sin \pi u \delta_{i+j, i'+j'-n} \delta_{j', n} \varepsilon(i' < n) \exp \left[\frac{-2\pi \sqrt{-1}}{n} (ju + 2i\eta) \right] \\
& + 4 \sin \pi u \sin \pi(u + 2\eta) \delta_{i+j, i'+j'-n} \delta_{i', n} \delta_{j', n} \exp \left[\frac{2\pi \sqrt{-1}}{n} \left(i - \frac{n}{2}\right)(u - 2\eta) \right].
\end{aligned} \tag{3.9}$$

The classical limit of this R -matrix agrees with the Belavin-Drinfeld classification of classical r -matrices [27].

The matrix elements with $i+j = i'+j'-n$ (the last 3 lines in (3.9)) can be eliminated by the gauge transformation with the diagonal matrix

$$D_{ij} = \delta_{ij} \exp [\pi(n-2j)\Lambda] \tag{3.10}$$

and subsequent limit $\Lambda \rightarrow -\infty$, so only the terms with $i+j = i'+j'$ survive. At the same time this gauge transformation does not change the form of the Yang-Baxter equation.

As a result, one obtains the R -matrix (appeared in [14], [8])

$$R(u)_{i'j'}^{ij} = \delta_{i,j} \delta_{i,i'} \delta_{i',j'} \frac{\sin \pi(u+2\eta)}{\sin 2\pi \eta}$$

$$\begin{aligned}
& + \delta_{i,i'} \delta_{j,j'} \varepsilon(i' \neq j') \frac{\sin \pi u}{\sin 2\pi \eta} \exp \left[\frac{2\pi \sqrt{-1} \eta}{n} (2(j-i) - n \operatorname{sign}(j-i)) \right] \\
& + \delta_{i,j'} \delta_{i',j} \varepsilon(i' \neq j') \exp \left[-\frac{\pi \sqrt{-1} u}{n} (2(j-i) - n \operatorname{sign}(j-i)) \right] \\
& - 2\sqrt{-1} \sin \pi u \delta_{i+j,i'+j'} \varepsilon(i' < i < j') \exp \left[\frac{2\pi \sqrt{-1}}{n} ((i-i')u + 2(j'-i)\eta) \right] \\
& + 2\sqrt{-1} \sin \pi u \delta_{i+j,i'+j'} \varepsilon(i' > i > j') \exp \left[\frac{2\pi \sqrt{-1}}{n} ((i-i')u + 2(j'-i)\eta) \right]. \quad (3.11)
\end{aligned}$$

Note that for $n = 2$ it coincides with the ordinary 6-vertex R -matrix.

4 The vertex-face correspondence for $n \geq 3$

Like in the case $n = 2$, matrix elements of the R -matrix (3.8) can be considered as Boltzmann weights of a lattice vertex model. We establish the correspondence of this vertex model with the trigonometric SOS model.

The trigonometric SOS model is an IRF-type model on a two-dimensional square lattice with statistical variables taking values in \mathbf{C}^n . Fix an orthonormal basis in \mathbf{C}^n : $\mathbf{C}^n = \oplus_{i=1,\dots,n} \mathbf{C} \epsilon_i$, $\langle \epsilon_i, \epsilon_j \rangle = \delta_{i,j}$ and let $\bar{\epsilon}_k = \epsilon_k - \frac{1}{n} \sum_{i=1}^n \epsilon_i$ for $k = 1, \dots, n$.

The Boltzmann weight corresponding to the configuration $\lambda, \mu, \mu', \nu \in \mathbf{C}^n$ round a face is denoted by

$$W \begin{bmatrix} & \mu & \\ \lambda & u & \nu \\ & \mu' & \end{bmatrix}.$$

The face weights for admissible configurations¹ are:

$$W \begin{bmatrix} & \lambda + 2\eta \bar{\epsilon}_r & \\ \lambda & u & \lambda + 4\eta \bar{\epsilon}_r \\ & \lambda + 2\eta \bar{\epsilon}_r & \end{bmatrix} = \frac{\sin \pi(u + 2\eta)}{\sin 2\pi \eta}, \quad (4.1)$$

$$W \begin{bmatrix} & \lambda + 2\eta \bar{\epsilon}_r & \\ \lambda & u & \lambda + 2\eta(\bar{\epsilon}_r + \bar{\epsilon}_s) \\ & \lambda + 2\eta \bar{\epsilon}_r & \end{bmatrix} = \frac{\sin \pi(-u + \lambda_{rs})}{\sin \pi \lambda_{rs}}, \quad r \neq s, \quad (4.2)$$

$$W \begin{bmatrix} & \lambda + 2\eta \bar{\epsilon}_r & \\ \lambda & u & \lambda + 2\eta(\bar{\epsilon}_r + \bar{\epsilon}_s) \\ & \lambda + 2\eta \bar{\epsilon}_s & \end{bmatrix} = \frac{\sin \pi u}{\sin 2\pi \eta} \frac{\sin \pi(2\eta + \lambda_{rs})}{\sin \pi \lambda_{rs}}, \quad r \neq s, \quad (4.3)$$

where $\lambda_{rs} \equiv \langle \lambda, \epsilon_r - \epsilon_s \rangle$. For all other configurations of λ, μ, μ', ν the face weight is put equal to 0. The weights (4.1)-(4.3) satisfy the star-triangle relation that is an analog of the Yang-Baxter equation for models of IRF type.

To get the vertex-face correspondence we introduce the trigonometric intertwining vectors $\tilde{\phi}(u)_\lambda^\mu$ with components $\tilde{\phi}(u)_\lambda^\mu_j$, $j = 1, \dots, n$. These vectors depend on $\lambda, \mu \in \mathbf{C}^n$ as well as on the spectral parameter u . The vectors are set to be zero unless $\mu - \lambda = 2\eta \bar{\epsilon}_k$. Components of the non-zero vectors are given by

$$\begin{aligned}
\tilde{\phi}(u)_\lambda^{\lambda+2\eta \bar{\epsilon}_k}_j &= \exp \left[\frac{\pi \sqrt{-1}}{n} (n-2j)(u - n \langle \lambda, \epsilon_k \rangle) \right], \quad j = 1, \dots, n-1, \\
\tilde{\phi}(u)_\lambda^{\lambda+2\eta \bar{\epsilon}_k}_n &= 2 \cos \pi(u - n \langle \lambda, \epsilon_k \rangle). \quad (4.4)
\end{aligned}$$

¹Usually the statistical variables of SOS models take values in the weight space of the $sl(n)$ algebra realized as the orthogonal complement to the vector $\sum_{i=1}^n \epsilon_i$ in \mathbf{C}^n . However, for our purposes it is more convenient to extend the definition of Boltzmann weights and intertwining vectors to the whole space \mathbf{C}^n . As it is seen from (4.1)-(4.3), (4.5), the selection rule for admissible weights and the vertex-face correspondence remain the same.

Then the trigonometric vertex-face correspondence holds in the form

$$\sum_{i,j=1}^n \tilde{R}(u-v)_{i',j'}^{i,j} \tilde{\phi}(u)_{\lambda i}^{\mu} \tilde{\phi}(v)_{\mu j}^{\nu} = \sum_{\mu'} \tilde{\phi}(v)_{\lambda j'}^{\mu'} \tilde{\phi}(u)_{\mu' i'}^{\nu} W \begin{bmatrix} \lambda & \mu & \nu \\ u-v & \mu' & \nu \end{bmatrix} \quad (4.5)$$

(see Appendix A). Here $\tilde{R}(u)_{i',j'}^{i,j}$ is the non-standard trigonometric R -matrix (3.8), (3.9).

Eq. (4.5) is the most general form of the trigonometric vertex-face correspondence. It admits certain simplifications. Let us shift the vector λ :

$$\lambda \rightarrow \lambda + \sqrt{-1}\Lambda \sum_{k=1}^n \epsilon_k,$$

where Λ is a constant. Then intertwining vectors (4.4) change as follows:

$$\tilde{\phi}(u)_{\lambda}^{\lambda+2\eta\bar{\epsilon}_k} \rightarrow \exp[\pi(n-2j)\Lambda] \tilde{\phi}(u)_{\lambda}^{\lambda+2\eta\bar{\epsilon}_k} \quad \text{for } j = 1, \dots, n-1,$$

$$\tilde{\phi}(u)_{\lambda}^{\lambda+2\eta\bar{\epsilon}_n} \rightarrow \exp(-\pi n\Lambda) \left\{ \exp[-\pi\sqrt{-1}(u-n < \lambda, \epsilon_k >)] + e^{2\pi n\Lambda} \exp[\pi\sqrt{-1}(u-n < \lambda, \epsilon_k >)] \right\}.$$

Let us apply to eq. (4.5) the gauge transformation with the diagonal matrix (3.10). Then the R -matrix $\tilde{R}(u)_{i',j'}^{ij}$ transforms as

$$\tilde{R}(u)_{i',j'}^{ij} \rightarrow \tilde{R}(u)_{i',j'}^{ij} \exp[2\pi(i' + j' - i - j)\Lambda].$$

Finally, take the limit $\Lambda \rightarrow -\infty$. The R -matrix $\tilde{R}(u)_{i',j'}^{ij}$ turns into $R(u)_{i',j'}^{ij}$ (3.11). The intertwining vectors (4.4) should be substituted by the simplified ones:

$$\phi(u)_{\lambda}^{\lambda+2\eta\bar{\epsilon}_k} = \exp\left[-\frac{2\pi j\sqrt{-1}}{n}(u-n < \lambda, \epsilon_k >)\right], \quad j = 1, \dots, n. \quad (4.6)$$

Eq.(4.5) acquires the form

$$\sum_{i,j=1}^n R(u-v)_{i',j'}^{i,j} \phi(u)_{\lambda i}^{\mu} \phi(v)_{\mu j}^{\nu} = \sum_{\mu'} \phi(v)_{\lambda j'}^{\mu'} \phi(u)_{\mu' i'}^{\nu} W \begin{bmatrix} \mu & \nu \\ u-v & \mu' \end{bmatrix}. \quad (4.7)$$

The vertex model is now associated with the simplified non-standard trigonometric R -matrix $R(u)_{i',j'}^{ij}$ (3.11) while the face model is the same as in eq. (4.5). The direct proof of formula (4.7) is given in Appendix B.

In the case $n = 2$ both the 7-vertex and 6-vertex models are connected with the same face model (4.1)-(4.3) via the vertex-face transformations (4.5) and (4.7), respectively. The intertwining vectors are different in the two cases. To avoid a confusion, we remark that in the paper [28] the 6-vertex R -matrix was related to a face model with *constant* (i.e., independent of λ) face weights by a transformation similar to (4.7). However, the intertwining vectors in that transformation differ from ours. In our scheme, that case corresponds to the *second* scaling limit discussed in Sect. 2 while our version (4.7) corresponds to the *first* one.

The intertwining vectors (4.6) can be simplified further. In fact their dependence on the spectral parameter is irrelevant. It can be eliminated by a gauge transformation. This is achieved by the transformation

$$R_{12}(u-v) \rightarrow A_1(u)A_2(v)R_{12}(u-v)(A_1(u)A_2(v))^{-1} := R'_{12}(u-v) \quad (4.8)$$

with the diagonal matrix

$$A(u)_{ij} = \delta_{ij} \exp\left[\frac{\pi\sqrt{-1}u}{n}(n-2j)\right].$$

Under this transformation the intertwining vectors lose their spectral parameters:

$$\phi(u)_{\lambda}^{\lambda+2\eta\bar{\epsilon}_k} \rightarrow \phi_{\lambda}^{\lambda+2\eta\bar{\epsilon}_k} := \exp[2\pi\sqrt{-1}j < \lambda, \epsilon_k >], \quad j = 1, \dots, n. \quad (4.9)$$

The R -matrix $R'(u)$ (4.8) has the form [14], [8]

$$\begin{aligned}
R'(u)_{i'j'}^{ij} &= \delta_{i,j} \delta_{i,i'} \delta_{i',j'} \frac{\sin \pi(u + 2\eta)}{\sin 2\pi\eta} \\
&+ \delta_{i,i'} \delta_{j,j'} \varepsilon(i' \neq j') \frac{\sin \pi u}{\sin 2\pi\eta} \exp \left[\frac{2\pi\sqrt{-1}\eta}{n} (2(j-i) - n \operatorname{sign}(j-i)) \right] \\
&+ \delta_{i,j'} \delta_{i',j} \varepsilon(i' \neq j') \exp \left[\operatorname{sign}(j-i) \pi \sqrt{-1} u \right] \\
&- 2\sqrt{-1} \sin \pi u \delta_{i+j,i'+j'} \varepsilon(i' < i < j') \exp \left[\frac{4\pi\sqrt{-1}\eta}{n} (j' - i) \right] \\
&+ 2\sqrt{-1} \sin \pi u \delta_{i+j,i'+j'} \varepsilon(i' > i > j') \exp \left[\frac{4\pi\sqrt{-1}\eta}{n} (j' - i) \right].
\end{aligned} \tag{4.10}$$

The vertex-face correspondence for intertwining vectors without spectral parameter reads

$$\sum_{i,j=1}^n R'(u)_{i'j'}^{i,j} \phi_{\lambda}^{\mu} \phi_{\mu j}^{\nu} = \sum_{\mu'} \phi_{\lambda}^{\mu'} \phi_{\mu' i'}^{\nu} W \begin{bmatrix} \mu \\ \lambda & u & \nu \\ & \mu' & \end{bmatrix}. \tag{4.11}$$

5 Factorized L -operator for the trigonometric Ruijsenaars model

Let us construct L -operators that satisfy the commutation relation

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v) \tag{5.1}$$

with the non-standard trigonometric R -matrices (3.8), (3.11) and (4.10). As soon as the intertwining vectors are known, this is straightforward. There is an important family of L -operators made out of the intertwining vectors. They are called *factorized L -operators* [5], [6]. The name comes from a factorized form in which they are presented. It turns out that the factorized L -operators serve as quantum Lax operators for the trigonometric Ruijsenaars model.

Let us begin with the most general trigonometric R -matrix (3.8), (3.9) and the corresponding intertwining vectors (4.4). There are n different vectors labeled by $\bar{\epsilon}_k$, $k = 1, \dots, n$. It is convenient to gather them in the $n \times n$ matrix $\tilde{\Phi}^\lambda(u)$ with matrix elements

$$(\tilde{\Phi}^\lambda(u))_{jk} = \tilde{\phi}(u)_\lambda^{\lambda+2\eta\bar{\epsilon}_k}_j. \tag{5.2}$$

The factorized L -operator reads

$$L^{(F)}(c|u) =: \tilde{\Phi}^\lambda(u+c) \cdot \hat{T} \cdot (\tilde{\Phi}^\lambda(u))^{-1} :. \tag{5.3}$$

Here $\tilde{\Phi}^{-1}$ denotes the inverse matrix, the dot means matrix product, c is a parameter and $\hat{T} = \operatorname{diag}(T_1, T_2, \dots, T_n)$ is the diagonal operator matrix whose matrix elements are shift operators: $T_k f(\lambda) = f(\lambda + 2\eta\bar{\epsilon}_k)$. The normal ordering $::$ means that the shift operators should be moved to the right after performing the matrix product (cf. (2.20)).

The L -operator (5.3) satisfies eq. (5.1) with the R -matrix (3.8) for any parameter $c \in \mathbf{C}$. The proof is based on the vertex-face correspondence (4.5) (cf. [1]).

It is useful to write down the factorized L -operator in a slightly decoded form. For that purpose, introduce components $\lambda_k = \langle \lambda, \epsilon_k \rangle$ of the vector λ . These very parameters are to be identified with coordinates of particles in the Ruijsenaars model. Matrix elements of $L^{(F)}(u)$ are difference operators acting to functions $f(\lambda_1, \dots, \lambda_n)$:

$$L_{ij}^{(F)}(c|u) f(\lambda_1, \dots, \lambda_n) = \sum_{k=1}^n (\tilde{\Phi}^\lambda(u+c))_{ik} (\tilde{\Phi}^\lambda(u))_{kj}^{-1} T_k f(\lambda_1, \dots, \lambda_n), \tag{5.4}$$

where

$$\begin{aligned} T_k f(\lambda_1, \dots, \lambda_n) \\ = f(\lambda_1 - \frac{2\eta}{n}, \dots, \lambda_{k-1} - \frac{2\eta}{n}, \lambda_k + \frac{2\eta(n-1)}{n}, \lambda_{k+1} - \frac{2\eta}{n}, \dots, \lambda_n - \frac{2\eta}{n}). \end{aligned}$$

Note that T_k commutes with the sum of coordinates $\sum \lambda_k$.

In a similar way, we find the factorized L -operator for the R -matrix (3.11). It is given by the same formula (5.3) with the matrix

$$(\Phi^\lambda(u))_{jk} = \phi(u)_\lambda^{\lambda+2\eta\bar{\epsilon}_k}_j \quad (5.5)$$

in place of $\tilde{\Phi}$. Here, the intertwining vectors are as in (4.6).

Finally, for the spectral parameter independent version (4.9) we introduce the matrix

$$\Phi_{jk}^\lambda = \phi_\lambda^{\lambda+2\eta\bar{\epsilon}_k}_j = \exp[2\pi\sqrt{-1}j \langle \lambda, \epsilon_k \rangle]. \quad (5.6)$$

The factorized L -operator takes the simple form

$$L =: \Phi^{\lambda-\kappa} \cdot \hat{T} \cdot (\Phi^\lambda)^{-1} :, \quad (5.7)$$

where $\kappa \equiv \frac{c}{n} \sum_{k=1}^n \epsilon_k$. It does not depend on the spectral parameter and satisfies

$$R'_{12}(u-v)L_1L_2 = L_2L_1R'_{12}(u-v) \quad (5.8)$$

with the R -matrix (4.10). These formulas generalize eqs (2.24)-(2.26) to the case $n \geq 3$.

Tending $u-v$ to $-\sqrt{-1}\infty$, it is possible to get rid of the spectral parameter in the R -matrix, too:

$$\begin{aligned} R_{CG}^{ij}{}_{i'j'} &= \delta_{i,j} \delta_{i,i'} \delta_{i',j'} \frac{\exp(2\pi\sqrt{-1}\eta)}{\sin 2\pi\eta} \\ &+ \delta_{i,i'} \delta_{j,j'} \varepsilon(i' \neq j') \frac{1}{\sin 2\pi\eta} \exp\left[\frac{2\pi\sqrt{-1}\eta}{n}(2(j-i) - n \operatorname{sign}(j-i))\right] \\ &+ \delta_{i,j'} \delta_{i',j} 2\sqrt{-1} \varepsilon(i' > j') \\ &- 2\sqrt{-1} \delta_{i+j,i'+j'} \varepsilon(i' < i < j') \exp\left[\frac{4\pi\sqrt{-1}\eta}{n}(j'-i)\right] \\ &+ 2\sqrt{-1} \delta_{i+j,i'+j'} \varepsilon(i' > i > j') \exp\left[\frac{4\pi\sqrt{-1}\eta}{n}(j'-i)\right]. \end{aligned} \quad (5.9)$$

This R -matrix coincides with the one given in (5.9). It satisfies the Yang-Baxter equation without spectral parameter [29]. For $n=2$, eq. (5.8) encodes commutation relations of the algebra of functions on the quantum group $SL(2)$.

Taking trace of any one of the constructed L -operators, we get, up to an irrelevant common factor,

$$M_1 = \sum_{k=1}^n \left(\prod_{j=1, j \neq k}^n \frac{\sin \pi(\lambda_{jk} + \frac{c}{n})}{\sin \pi \lambda_{jk}} \right) T_k. \quad (5.10)$$

This operator is gauge equivalent to the first non-trivial Hamiltonian of the trigonometric n -particle Ruijsenaars model.

Applying the fusion procedure to the factorized L -operators, it is possible to obtain the whole commutative family of IM as traces of the fused L -operators (see [1] for details). In this way one gets the commuting Macdonald operators [30]

$$M_d = \sum_{I \subset \{1, \dots, n\}, |I|=d} \left(\prod_{r \notin I, s \in I} \frac{\sin \pi(\lambda_{rs} + \frac{c}{n})}{\sin \pi \lambda_{rs}} \right) T_I, \quad d = 1, \dots, n-1, \quad (5.11)$$

where $T_I = \prod_{i \in I} T_i$. By a conjugation with a function of λ they yield higher commuting Hamiltonians of the trigonometric Ruijsenaars model.

The generating function for these IM is

$$: \det[L - z] := \sum_{d=0}^n (-z)^{n-d} M_d, \quad (5.12)$$

where $M_0 = 1$, $M_n = \prod_{k=1}^n T_k$. In the elliptic case, a similar formula was proved in [1].

6 Connection with the dynamical R -matrix approach

A more familiar approach to the Ruijsenaars-like models is based on dynamical R -matrices. Let us show how to get the dynamical R -matrix [22], [8]

$$\begin{aligned} R^D(u, \lambda)_{r's}^{r's'} &= \delta_{rr'} \delta_{ss'} \delta_{rs} \frac{\sin \pi(u + 2\eta)}{\sin 2\pi\eta} \\ &+ \delta_{rs'} \delta_{r's} \varepsilon(r \neq s) \frac{\sin \pi(-u + \lambda_{rs})}{\sin \pi \lambda_{rs}} \\ &+ \delta_{rr'} \delta_{ss'} \varepsilon(r \neq s) \frac{\sin \pi u}{\sin 2\pi\eta} \frac{\sin \pi(2\eta + \lambda_{rs})}{\sin \pi \lambda_{rs}} \end{aligned} \quad (6.1)$$

from the non-standard trigonometric one (3.11) by a quasi-Hopf twist [31].

Consider the vertex-face correspondence (4.7) with the face weights (4.1)-(4.3) and the R -matrix (3.11). By $\Phi_1^\lambda(u)$ denote the matrix of intertwining vectors (5.5) in the first copy of the space \mathbf{C}^n :

$$(\Phi^\lambda(u))_{i'}^i = \phi(u)_\lambda^{\lambda+2\eta\bar{\epsilon}_i}_{i'}$$

and by $\Psi_{12}^\lambda(u)$ denote the following matrix in the tensor product $\mathbf{C}^n \times \mathbf{C}^n$:

$$(\Psi^\lambda(u))_{i'j'}^{ij} = \delta_{ii'} \phi(u)_{\lambda+2\eta\bar{\epsilon}_{i'}}^{\lambda+2\eta\bar{\epsilon}_i+2\eta\bar{\epsilon}_j}_{j'}$$

(it is a diagonal matrix in the first copy of \mathbf{C}^n).

In this notation, the vertex-face correspondence (4.7) can be written as a matrix equation:

$$R_{12}(u-v) \Phi_1^\lambda(u) \Psi_{12}^\lambda(v) = \Phi_2^\lambda(v) \Psi_{21}^\lambda(u) W_{12}(u-v, \lambda) \quad (6.2)$$

For convenience, we write down the same equation with indices:

$$\sum_{i,j,k} R(u-v)_{i'j'}^{ij} (\Phi^\lambda(u))_i^k (\Psi^\lambda(v))_{kj}^{mr} = \sum_{i,j,r'} (\Phi^\lambda(v))_{j'}^{r'} (\Psi^\lambda(u))_{i'r'}^{ij} W(u-v, \lambda)_{ij}^{mr}. \quad (6.3)$$

Here the matrix $W_{12}(u, \lambda)$ is

$$(W(u, \lambda))_{r's'}^{r's} = \delta_{r+s, r'+s'} W \begin{bmatrix} \lambda + 2\eta\bar{\epsilon}_r & & \\ \lambda & u & \lambda + 2\eta(\bar{\epsilon}_r + \bar{\epsilon}_s) \\ & \lambda + 2\eta\bar{\epsilon}_{s'} & \end{bmatrix}$$

which coincides with the dynamical R -matrix (6.1):

$$W_{12}(u, \lambda) = R_{12}^D(u, \lambda). \quad (6.4)$$

The W -weights are matrix elements of the dynamical R -matrix.

We conclude that the dynamical and non-dynamical R -matrices are connected by a quasi-Hopf twist:

$$R_{12}(u-v) F_{12}(u, v; \lambda) = F_{21}(v, u; \lambda) R_{12}^D(u-v, \lambda), \quad (6.5)$$

where

$$F_{12}(u, v; \lambda) = \Phi_1^\lambda(u) \Psi_{12}^\lambda(v). \quad (6.6)$$

Thus we have represented the vertex-face correspondence (4.7) in the form of a quasi-Hopf twist connecting the non-dynamical R -matrix with a dynamical one². The matrix F_{12} in the form (6.6) (up to a factor commutative with the R -matrix) was calculated for $n = 2$ in [32].

It is known [8] that the dynamical R -matrix (6.1) is also related to the standard trigonometric R -matrix (3.4) by another quasi-Hopf twist:

$$R_{A_{n-1}}(u)_{12} \tilde{F}_{12}(\lambda) = \tilde{F}_{21}(\lambda) R_{12}^D(u, \lambda), \quad (6.7)$$

$$\begin{aligned} \tilde{F}(\lambda)_{r's'}^{r's} &= 2\sqrt{-1} \delta_{rr'} \delta_{ss'} \delta_{rs} \\ &+ \delta_{rr'} \delta_{ss'} \varepsilon(r < s) \frac{1}{\sin \pi \lambda_{rs}}, \\ &+ \delta_{rr'} \delta_{ss'} \varepsilon(r > s) \frac{1}{\sin \pi (2\eta - \lambda_{rs})}, \\ &- \delta_{rs'} \delta_{r's} \varepsilon(r < s) \frac{\exp[\pi \sqrt{-1} \lambda_{rs}] \sin 2\pi \eta}{\sin \pi \lambda_{rs} \sin \pi (2\eta + \lambda_{rs})} \end{aligned}$$

Comparing the relations (6.5) and (6.7), one has

$$R_{12}(u - v) = F'_{21}(v, u; \lambda) R_{A_{n-1}}(u - v)_{12} (F'_{12}(u, v; \lambda))^{-1} \quad (6.8)$$

with

$$F'_{12}(u, v; \lambda) = F_{12}(u, v; \lambda) (\tilde{F}_{12}(\lambda))^{-1}.$$

Therefore, the two *non-dynamical* R -matrices turn out to be related by a quasi-Hopf twist which depends on the dynamical variables.

7 Conclusion

Let us summarize the results. Starting from the elliptic $n^2 \times n^2$ Belavin R -matrix $R^{(ell)}(u)$, we have considered the chain

$$\tilde{R}(u) \rightarrow R(u) \rightarrow R'(u) \rightarrow R_{CG}$$

of *non-standard* trigonometric R -matrices obtained as its different degenerations and given by (3.8), (3.11), (4.10) and (5.9), respectively. We call them non-standard because they differ from the standard trigonometric R -matrix $R_{A_{n-1}}(u)$ (3.4). The arrows mean certain types of degeneration procedures described in the main body of the paper. The last R -matrix in this chain does not depend on the spectral parameter and coincides with the constant R -matrix introduced by Cremmer and Gervais [9] in another context.

There are two things common for all these R -matrices:

- They satisfy the standard Yang-Baxter equation
- They are *non-dynamical* R -matrices for the n -particle trigonometric Ruijsenaars model.

The quantum Lax matrices for the trigonometric Ruijsenaars model intertwined by these R -matrices have been constructed using the technique of intertwining vectors. In each case, the Hamiltonian of the model is obtained as trace of the Lax matrix up to a common non-essential factor.

It is surprising that the standard trigonometric R -matrix $R_{A_{n-1}}(u)$ *does not* belong to this chain (except for the case $n = 2$) in the sense that it is not appropriate for the Ruijsenaars-type models.

²The quasi-Hopf relations (6.5), (6.6) are also valid for the vertex-face correspondence (4.5) with non-standard trigonometric R -matrix (3.8), (3.9) and the intertwining vectors (4.4).

Though, it is obtained as a limiting case of the Belavin R -matrix, too, in a way around the chain. It has been also shown that the non-standard R -matrices (3.8), (3.11) can be obtained from the standard one (3.4) by a quasi-Hopf twist.

A more customary (and until the very recent time the only available) R -matrix formulation of the Ruijsenaars-type models is based on *dynamical* R -matrices. The explicit connection between the dynamical and non-dynamical R -matrices is given by eq. (6.5) that has the form of Drinfeld's quasi-Hopf twist. At the same time this is a simple reformulation of the famous vertex-face correspondence between "vertex" and "face" type lattice statistical models with trigonometric Boltzmann weights. The dynamical R -matrix is identified with the matrix of W -weights for the face model.

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Appendix A

In this Appendix we show how to get the twisted trigonometric limit of the intertwining vectors. We use the theta functions with rational characteristics (3.1). We also need the Jacobi theta function

$$\theta_1(u) = \sum_{m \in \mathbf{Z}} \exp \left(\pi i \tau \left(m + \frac{1}{2} \right)^2 + 2\pi i \left(m + \frac{1}{2} \right) \left(u + \frac{1}{2} \right) \right).$$

The elliptic face weights are [17], [5]:

$$\begin{aligned} W^{(ell)} \begin{bmatrix} \lambda + 2\eta\bar{\epsilon}_i & & \\ \lambda & u & \lambda + 4\eta\bar{\epsilon}_i \\ & \lambda + 2\eta\bar{\epsilon}_i & \end{bmatrix} &= \frac{\theta_1(u + 2\eta)}{\theta_1(2\eta)}, \\ W^{(ell)} \begin{bmatrix} \lambda + 2\eta\bar{\epsilon}_i & & \\ \lambda & u & \lambda + 2\eta(\bar{\epsilon}_i + \bar{\epsilon}_j) \\ & \lambda + 2\eta\bar{\epsilon}_i & \end{bmatrix} &= \frac{\theta_1(-u + \lambda_{ij})}{\theta_1(\lambda_{ij})}, \quad i \neq j, \\ W^{(ell)} \begin{bmatrix} \lambda + 2\eta\bar{\epsilon}_i & & \\ \lambda & u & \lambda + 2\eta(\bar{\epsilon}_i + \bar{\epsilon}_j) \\ & \lambda + 2\eta\bar{\epsilon}_j & \end{bmatrix} &= \frac{\theta_1(u)}{\theta_1(2\eta)} \frac{\theta_1(2\eta + \lambda_{ij})}{\theta_1(\lambda_{ij})}, \quad i \neq j. \end{aligned}$$

The vertex-face correspondence is implemented by elliptic intertwining vectors

$$\phi^{(ell)}(u)_{\lambda j}^\mu = \begin{cases} \theta^j(u - n < \lambda, \bar{\epsilon}_k > + \frac{n-1}{2}) & : \mu - \lambda = 2\eta\bar{\epsilon}_k \quad \text{for some } k = 1, \dots, n, \\ 0 & : \text{otherwise} \end{cases}$$

The elliptic vertex-face correspondence has the form

$$\begin{aligned} &\sum_{i,j=1}^n R^{(ell)}(u-v)_{i'j'}^{i,j} \phi^{(ell)}(u)_{\lambda i}^\mu \phi^{(ell)}(v)_{\mu j}^\nu \\ &= \rho(u) \sum_{\mu'} \phi^{(ell)}(v)_{\lambda j'}^{\mu'} \phi^{(ell)}(u)_{\mu' i'}^\nu W^{(ell)} \begin{bmatrix} \mu & & \\ \lambda & u-v & \nu \\ & \mu' & \end{bmatrix}, \end{aligned}$$

where $\rho(u)$ is a normalization factor such that $\rho(u) \rightarrow 1$ as $h \rightarrow 0$, $h = e^{\pi\sqrt{-1}\tau}$.

Taking the limit or $h \rightarrow 0$ of this equation and using formulas given in Sect. 3, we see that intertwining vectors diverge. To regularize them, let us extract the singular h -dependent factors:

$$\phi^{(ell)}(u)_\mu^{\mu+2\eta\bar{\epsilon}_k j} = h^{n(\frac{1}{2}-\frac{j}{n})^2} \varphi^{(ell)}(u)_\mu^{\mu+2\eta\bar{\epsilon}_k j} \quad \text{for } j = 1, \dots, n,$$

where $\varphi^{(ell)}(u)_\mu^{\mu+2\eta\bar{\epsilon}_k j}$ already have a smooth limit $h \rightarrow 0$ coinciding with vectors $\tilde{\phi}(u)_\mu^{\mu+2\eta\bar{\epsilon}_k j} + \sum_{\alpha>0} h^\alpha \phi_\alpha$ introduced in eq. (4.4). Rewriting the elliptic vertex-face correspondence as

$$\begin{aligned} & \sum_{i,j=1}^n \left(R^{(ell)}(u-v)_{i',j}^{i,j} h^{-n(\frac{1}{2}-\frac{j'}{n})^2 - n(\frac{1}{2}-\frac{j}{n})^2 + n(\frac{1}{2}-\frac{j}{n})^2 + n(\frac{1}{2}-\frac{j'}{n})^2} \right) \varphi^{(ell)}(u)_\lambda^\mu \varphi^{(ell)}(v)_{\mu j}^\nu \\ &= \rho(u) \sum_{\mu'} \varphi^{(ell)}(v)_\lambda^{\mu'} \varphi^{(ell)}(u)_{\mu' i'}^\nu W^{(ell)} \begin{bmatrix} \mu & & \\ \lambda & u-v & \nu \\ & \mu' & \end{bmatrix} \end{aligned}$$

and taking the limit, we come to the trigonometric version (4.5).

Appendix B

Here we give a direct proof of eq. (4.11).

Specify the face variables as

$$\mu = \lambda + 2\eta\bar{\epsilon}_r \text{ and } \nu = \mu + 2\eta\bar{\epsilon}_s \quad \text{for some } r, s = 1, \dots, n.$$

One starts with the case $i' = j'$. The R -matrix in the l.h.s. of (4.11) is equal to

$$\delta_{i,j} \delta_{i,i'} \delta_{i',j'} \frac{\sin \pi(u+2\eta)}{\sin 2\pi\eta}.$$

Let $r = s$. Then we have the only term in the r.h.s. with the W -weight (4.1). The vertex-face correspondence (4.11) is evident in this case.

Now let $r \neq s$. The l.h.s. of (4.11) reads

$$\frac{\sin \pi(u+2\eta)}{\sin 2\pi\eta} \exp \left[2\pi\sqrt{-1}j' \left(< \lambda, \epsilon_s + \epsilon_r > - \frac{2\eta}{n} \right) \right].$$

In the r.h.s. one has two terms corresponding to W -weights (4.2) and (4.3):

$$\begin{aligned} & \frac{\sin \pi(-u + \lambda_{rs})}{\sin \pi\lambda_{rs}} \exp \left[2\pi\sqrt{-1}j' \left(< \lambda, \epsilon_s + \epsilon_r > - \frac{2\eta}{n} \right) \right] \\ & + \frac{\sin \pi u}{\sin 2\pi\eta} \frac{\sin \pi(2\eta + \lambda_{rs})}{\sin \pi\lambda_{rs}} \exp \left[2\pi\sqrt{-1}j' \left(< \lambda, \epsilon_s + \epsilon_r > - \frac{2\eta}{n} \right) \right]. \end{aligned}$$

Eq. (4.11) follows from the trigonometric identity

$$\frac{\sin \pi(u+2\eta)}{\sin 2\pi\eta} = \frac{\sin \pi(-u + \lambda_{rs})}{\sin \pi\lambda_{rs}} + \frac{\sin \pi(2\eta + \lambda_{rs})}{\sin \pi\lambda_{rs}}.$$

Let $i' < j'$. Then the R -matrix in the l.h.s. of (4.11) takes the form

$$\begin{aligned} & \delta_{i,i'} \delta_{j,j'} \frac{\sin \pi u}{\sin 2\pi\eta} \exp 2\pi\sqrt{-1}\eta \left[\frac{2}{n}(j' - i') - 1 \right] \\ & + \delta_{i,j'} \delta_{i',j} \exp \left[-\pi\sqrt{-1}u \right] \\ & - 2 \sin \pi u \delta_{i+j,i'+j'} \varepsilon(i' < i < j') \exp 2\pi\sqrt{-1} \left[\frac{1}{4} + \frac{2\eta}{n}(j' - i) \right]. \end{aligned}$$

The l.h.s. of (4.11) reads

$$\begin{aligned} & \frac{\sin \pi u}{\sin 2\pi \eta} \exp 2\pi \sqrt{-1} \eta \left[\frac{2}{n} (j' - i') - 1 \right] \phi_{\lambda i'}^{\mu} \phi_{\mu j'}^{\nu} + \exp [-\pi \sqrt{-1} u] \phi_{\lambda j'}^{\mu} \phi_{\mu i'}^{\nu} \\ & - 2 \sin \pi u \sum_{1 \leq i' < i < j' \leq n} \exp 2\pi \sqrt{-1} \left[\frac{1}{4} + \frac{2\eta}{n} (j' - i) \right] \phi_{\lambda i}^{\mu} \phi_{\mu i' + j' - i}^{\nu}. \end{aligned}$$

After some algebra the l.h.s. becomes

$$\begin{aligned} \text{l.h.s} &= \frac{\sin \pi u}{\sin 2\pi \eta} \exp 2\pi \sqrt{-1} \left[i' (< \lambda, \epsilon_r > - \frac{2\eta}{n}) + j' (< \lambda, \epsilon_s > + 2\eta \delta_{rs}) - \eta \right] \\ &+ \exp \pi \sqrt{-1} (-u) \exp 2\pi \sqrt{-1} \left[j' < \lambda, \epsilon_r > + i' (< \lambda, \epsilon_s > + 2\eta (\delta_{rs} - \frac{1}{n})) \right] \\ &- 2 \sin \pi u \exp 2\pi \sqrt{-1} \left[\frac{1}{4} + i' (< \lambda, \epsilon_s > + 2\eta (\delta_{rs} - \frac{1}{n})) + j' (< \lambda, \epsilon_s > + 2\eta \delta_{rs}) \right] \\ &\times \frac{\exp 2\pi \sqrt{-1} [(\lambda_{rs} - 2\eta \delta_{rs})(i' + 1)] - \exp 2\pi \sqrt{-1} [(\lambda_{rs} - 2\eta \delta_{rs}) j']}{1 - \exp 2\pi \sqrt{-1} (\lambda_{rs} - 2\eta \delta_{rs})}. \end{aligned}$$

Now specify the indexes r, s . First, let $r = s$, then the l.h.s. is

$$\begin{aligned} \text{l.h.s} &= \exp 2\pi \sqrt{-1} \left[i' (< \lambda, \epsilon_r > - \frac{2\eta}{n}) + j' < \lambda, \epsilon_r > \right] \\ &\times \left\{ \frac{\sin \pi u}{\sin 2\pi \eta} \exp 2\pi \sqrt{-1} [2j' \eta - \eta] + \exp \pi \sqrt{-1} (-u) \exp [4\pi \eta \sqrt{-1} i'] \right. \\ &- \left. \frac{\sin \pi u}{\sin 2\pi \eta} \exp 2\pi \sqrt{-1} [2i' \eta + 2j' \eta] (\exp 2\pi \sqrt{-1} [-2i' \eta - \eta] - \exp 2\pi \sqrt{-1} [-2j' \eta + \eta]) \right\} \end{aligned}$$

In the case $r = s$ there is only one term in the r.h.s. of (4.11) corresponding to the face weight (4.1):

$$\text{r.h.s} = \exp 2\pi \sqrt{-1} \left[i' (< \lambda, \epsilon_r > + 2\eta (1 - \frac{1}{n})) + j' < \lambda, \epsilon_r > \right] \frac{\sin \pi (u + 2\eta)}{\sin \pi (2\eta)}.$$

The equality of the l.h.s. and the r.h.s. follows from the identity

$$\exp [-\pi \sqrt{-1} u] + \frac{\sin \pi u}{\sin 2\pi \eta} \exp [2\pi \sqrt{-1} \eta] = \frac{\sin \pi (u + 2\eta)}{\sin 2\pi \eta}.$$

Consider now the case $r \neq s$, then the l.h.s. of eq. (4.11) after some simplifications reads

$$\begin{aligned} \text{l.h.s} &= \frac{\sin \pi u}{\sin 2\pi \eta} \exp 2\pi \sqrt{-1} \left[i' (< \lambda, \epsilon_r > - \frac{2\eta}{n}) + j' < \lambda, \epsilon_s > - \eta \right] \\ &+ \exp \pi \sqrt{-1} (-u) \exp 2\pi \sqrt{-1} \left[j' < \lambda, \epsilon_r > + i' (< \lambda, \epsilon_s > - \frac{2\eta}{n}) \right] \\ &- \frac{\sin \pi u}{\sin \pi \lambda_{rs}} \exp 2\pi \sqrt{-1} \left[(i' + \frac{1}{2}) < \lambda, \epsilon_s > - \frac{2i' \eta}{n} + (j' - \frac{1}{2}) < \lambda, \epsilon_r > \right] \\ &+ \frac{\sin \pi u}{\sin \pi \lambda_{rs}} \exp 2\pi \sqrt{-1} \left[(i' + \frac{1}{2}) < \lambda, \epsilon_r > - \frac{2i' \eta}{n} + (j' - \frac{1}{2}) < \lambda, \epsilon_s > \right] \end{aligned}$$

In the r.h.s. there are two terms corresponding to the W -weights (4.2) and (4.3):

$$\begin{aligned} \text{r.h.s} &= \exp 2\pi \sqrt{-1} \left[i' < \lambda, \epsilon_s > - \frac{2i' \eta}{n} + j' < \lambda, \epsilon_r > \right] \frac{\sin \pi (-u + \lambda_{rs})}{\sin \pi \lambda_{rs}} \\ &+ \exp 2\pi \sqrt{-1} \left[i' < \lambda, \epsilon_r > - \frac{2i' \eta}{n} + j' < \lambda, \epsilon_s > \right] \frac{\sin \pi u}{\sin 2\pi \eta} \frac{\sin \pi (2\eta + \lambda_{rs})}{\sin \pi \lambda_{rs}}. \end{aligned}$$

It is easy to see that the vertex-face correspondence in this case is equivalent to

$$A \exp 2\pi\sqrt{-1} [i' < \lambda, \epsilon_s > + j' < \lambda, \epsilon_r >] + B \exp 2\pi\sqrt{-1} [i' < \lambda, \epsilon_r > + j' < \lambda, \epsilon_s >] \equiv 0,$$

where A and B are some trigonometric functions of η , u and λ_{rs} . Thus, to prove the vertex-face correspondence it is enough to show that $A \equiv 0$ and $B \equiv 0$. The result follows from

$$\frac{\exp[-2\pi\sqrt{-1}\eta]}{\sin 2\pi\eta} + \frac{\exp[\pi\sqrt{-1}\lambda_{rs}]}{\sin \pi\lambda_{rs}} - \frac{\sin \pi(2\eta + \lambda_{rs})}{\sin 2\pi\eta \sin \pi\lambda_{rs}} \equiv 0$$

and

$$-\exp[-\pi\sqrt{-1}u] + \frac{\sin \pi u}{\sin \pi\lambda_{rs}} \exp[-\pi\sqrt{-1}\lambda_{rs}] + \frac{\sin \pi(-u + \lambda_{rs})}{\sin \pi\lambda_{rs}} \equiv 0.$$

The case $i' > j'$ is considered similarly.

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